Lecture 14 on Oct. 31 2013

In the last lecture note, we showed that for $f(z) = z$ and $g(z) \equiv 1$, the integrals over unit circle are 0. But the integral of $1/z$ over the unit circle is $2\pi i$. In this lecture, we consider the following question:

Q: Given Ω an open set in $\mathbb{R}^2$, when can we have

$$\int_\gamma f(z)\,\mathrm{d}z = 0, \qquad \text{for all } \gamma \text{ a closed curve in } \Omega.$$

Supposing that $f = u + iv$, by definition, we know that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_\gamma v\,\mathrm{d}y + u\,\mathrm{d}x.$$

Therefore the above question is reduced to find u and v so that

$$\int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y = 0 \qquad \text{and} \qquad \int_\gamma v\,\mathrm{d}y + u\,\mathrm{d}x = 0, \quad \text{for all } \gamma \text{ a closed curve in } \Omega.$$

Standard arguments from calculus show that there must have F_1 and F_2 so that

$$\partial_x F_1 = u, \qquad \partial_y F_1 = -v, \qquad \partial_x F_2 = v, \qquad \partial_y F_2 = u.$$

Setting $F = F_1 + iF_2$, the above equalities tell us that F satisfies Cauchy Riemann equation and meanwhile $F' = \partial_x F_1 + i\partial_x F_2 = u + iv = f$. Therefore f must be a derivative of some analytic function. Moreover if $f = F'$ where F is an analytic function on Ω, then the integral in the above question is 0 for all γ a closed curve in Ω. Now let us take a look at the three examples in the last lecture. $f(z) = z$ is the derivative of $F(z) = z^2/2$. $g(z) \equiv 1$ is the derivative of $F(z) = z$. Therefore the integral of f and g over any closed curve must be 0. The function $h(z) = 1/z$ is different. We know that it must be the derivative of $\log z$. But $\log z$ must be defined on $\mathbb{C} \setminus l$, where l is a continuous curve connecting 0 and ∞. If γ is the unit circle, we know that for any open set containing γ, l must have an intersection with this open set. Therefore $\log z$ is not analytic for all points in the open set. So we can not imply that $\int_\gamma 1/z\mathrm{d}z = 0$ for all γ a closed curve. In fact the third example in the last lecture shows that the integral of $1/z$ over the unit circle equals to $2\pi i$.